UDC 539.3

CONSTRUCTION OF GREEN'S FUNCTIONS AND MATRICES FOR EQUATIONS AND SYSTEMS OF THE ELLIPTIC TYPE

PMM Vol. 42, No. 4, 1978, pp. 695-700

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(Received November 18, 1977)

A method of constructing Green's functions and matrices of mixed boundary value problems for regions bounded by coordinate lines in a rectangular Cartesian and polar coordinate systems (band, half-plane, rectangle, circle, ring, and circle and ring sectors) is presented. Several closed representations of such functions and matrices are obtained for Laplace equations and Lamé's system for the plane problem of the theory of elasticity.

1. Let us determined the solution $\mathbf{U} = \mathbf{U}(x, y)$ of the plane problem of the theory of elasticity defined by displacements

$$
\begin{aligned}
&L(\partial^2 / \partial x^2, \partial^2 / \partial y^2, \lambda, \mu)\, \mathbf{U}(x, y) = \mathbf{F}(x, y) \qquad (1.1)\\
&B_1(\partial / \partial x, \partial / \partial y)\, \mathbf{U}(x, 0) = B_2(\partial / \partial x, \partial / \partial y)\, \mathbf{U}(x, b) = 0\\
&B_3(\partial / \partial x, \partial / \partial y)\, \mathbf{U}(0, y) = B_4(\partial / \partial x, \partial / \partial y)\, \mathbf{U}(a, y) = 0
\end{aligned}
$$

in the rectangle $\Omega\ (0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b)$. In these equations $\mathbf{U}(x, y)$ and $\mathbf{F}(x, y)$ are vectors of displacements of points of rectangle Ω and of volume forces, respectively; λ and μ are Lamé's constants that define the elastic properties of the material filling the Ω-space. The elements of the matrix operator $L = (L_{ij})_{2;2}$ are determined by formulas

$$
\begin{aligned}
&L_{11} \equiv (\lambda + \mu)\partial^2 / \partial x^2 + \Delta, \quad L_{12} = L_{21} \equiv (\lambda + \mu)\, \partial^2 / \partial x\ \partial y\\
&I_{22} \equiv (\lambda + \mu)\partial^2 / \partial y^2 + \Delta
\end{aligned}
$$

Operators $B_i\ (i = 1, \ldots, 4)$ define the conditions of interaction of the considered rectangle with surrounding medium.

We assume that the solution of problem (1.1) and vector $\mathbf{F}(x, y)$ can be represented by the expansions

$$
\mathbf{U}(x, y) = \sum_{n=0}^{\infty} Q_n(y)\, \mathbf{U}_n(x), \quad \mathbf{F}(x, y) = \sum_{n=0}^{\infty} Q_n(y)\, \mathbf{F}_n(x) \qquad (1.2)
$$

$$
Q_n(y) = \left\| \begin{matrix} \cos \nu y & 0 \\ 0 & \sin \nu y \end{matrix} \right\|, \quad \nu = n\pi b^{-1}
$$

Note that such representation must ensure the fulfilment of the first two boundary conditions of problem (1.1), which, for instance, occurs in the case, which is of practical interest, of the stress-strain state symmetry with respect to sides $y = 0$ and

$y = b$ of the Ω-rectangle when

$$B_1 = B_2 \equiv \left\| \begin{matrix} 0 & I \\ \mu\, \partial / \partial y & \mu\, \partial / \partial x \end{matrix} \right\| \tag{1.3}$$

In that case we obtain from (1.1) and (1.2) the system of ordinary differential equations

$$L_n\,(\partial^2 / \partial x^2, \lambda, \mu)\, \mathbf{U}_n(x) = \mathbf{F}_n(x) \quad (n = 0, 1, 2, \ldots) \tag{1.4}$$

with boundary conditions

$$B_{3n}\,(\partial / \partial x)\, \mathbf{U}_n(0) = B_{4n}\,(\partial / \partial x)\, \mathbf{U}_n(a) = 0 \tag{1.5}$$

Elements L_{ij}^n of matrix L_n are determined by formulas

$$L_{11}^n \equiv (\lambda + 2\mu)\, \partial^2 / \partial x^2 - \nu^2\mu, \quad L_{12}{}^n = - L_{21}{}^n \equiv (\lambda + \mu)\, \nu \partial / \partial x$$
$$L_{22}{}^n \equiv \mu\, \partial^2 / \partial x^2 - (\lambda + 2\mu)\, \nu^2$$

Vectors

$$\mathbf{U}_{n1}(x) = \left\| \begin{matrix} \exp \nu x \\ -\exp \nu x \end{matrix} \right\|, \quad \mathbf{U}_{n3}(x) = \left\| \begin{matrix} -(\lambda + \mu)\, \nu x \exp \nu x \\ [(\lambda + \mu)\, \nu x + (\lambda + 3\mu)] \exp \nu x \end{matrix} \right\| \tag{1.6}$$
$$\mathbf{U}_{n2}(x) = \left\| \begin{matrix} \exp(-\nu x) \\ \exp(-\nu x) \end{matrix} \right\|, \quad \mathbf{U}_{n4}(x) = \left\| \begin{matrix} (\lambda + \mu)\, \nu x \exp(-\nu x) \\ [(\lambda + \mu)\nu x - (\lambda + 3\mu)] \exp(-\nu x) \end{matrix} \right\|$$

represent the fundamental system of solutions of the homogeneous system that corresponds to (1.4) for $n = 1, 2, 3, \ldots$ (the case of $n = 0$, although trivial, must be considered separately).

Using the procedure of the Lagrange method of varying independent variables, we obtain the general solution of system (1.4) of the form

$$\mathbf{U}_n(x) = \int_0^x S_n(x, \xi)\, \mathbf{F}_n(\xi)\, d\xi + P_n(x)\, \mathbf{D}_n \tag{1.7}$$

where the elements $S_{ij}{}^n(x, \xi)$ of matrix $S_n(x, \xi)$ are determined by formulas

$$S_{11}{}^n(x, \xi) = a(x - \xi) - b(x - \xi), \quad S_{12}{}^n(x, \xi) = -mc(x - \xi)$$
$$S_{21}{}^n(x, \xi) = c(x - \xi), \quad S_{22}{}^n(x, \xi) = m\,[a(x - \xi) + b(x - \xi)]$$
$$a(u) = {}^1/_2 \nu^{-1} (\lambda + 3\mu)\, \mathrm{sh}\, \nu u, \quad b(u) = {}^1/_2 (\lambda + \mu)\, u\, \mathrm{ch}\, \nu u$$
$$c(u) = {}^1/_2 (\lambda + \mu)\, u\, \mathrm{sh}\, \nu u, \quad m = \mu\, (\lambda + 2\mu)^{-1}$$

and $P_n(x) = (\mathbf{U}_{nj}(x))$ is a 2×4 matrix whose columns are represented by vectors (1.6).

The column matrix of arbitrary constants D_n must satisfy formula (1.7) and boundary conditions (1.5), and is determined by the integral

$$\mathbf{D}_n = \int_0^a W_n(\xi)\,\mathbf{F}_n(\xi)\,d\xi$$

which after substitution into (1.7) yields

$$\mathbf{U}_n(x) = \int_0^a g_n(x, \xi)\,\mathbf{F}_n(\xi)\,d\zeta \quad (n = 1, 2, 3, \ldots) \tag{1.8}$$

The kernel

$$g_n(x, \xi) = \begin{cases} S_n(x, \xi) + P_n(x) \quad W_n(\xi) & \text{for} \quad x \geqslant \xi \\ \qquad\qquad\quad P_n(x) \quad W_n(\xi) & \text{for} \quad x \leqslant \xi \end{cases}$$

of integral (1.8) is the Green's matrix of the boundary value problem (1.4), (1.5). As already mentioned, $g_0(x, \xi)$ can also be obtained by the described method using the fundamental system of solutions for $n = 0$.

Applying now the Fourier-Euler transformation formula to $\mathbf{F}_n(\xi)$, from (1.8) and the first of formulas (1.2) we obtain

$$\mathbf{U}(x, y) = \int_0^a\int_0^b \left[\frac{\varepsilon_n}{b}\sum_{n=0}^{\infty} Q_n(y)\,g_n(x, \xi)\,Q_n(\eta)\right] \mathbf{F}(\xi, \eta)\,d\xi\,d\eta \tag{1.9}$$

$$\varepsilon_n = \begin{cases} 1 & \text{for} \quad n = 0 \\ 2 & \text{for} \quad n > 0 \end{cases}$$

Owing to the uniqueness of solution of problem (1.1) and the known corollary (see, e.g., [1]) from the second Green's formula the kernel of integral (1.9)

$$G(x, y; \xi, \eta) = \frac{\varepsilon_n}{b}\sum_{n=0}^{\infty} Q_n(y)\,g_n(x, \xi)\,Q_n(\eta) \tag{1.10}$$

is the sought Green's matrix of this problem.

In some cases it is possible to summate expressions of the type (1.10). Thus, for example, by formulating problem (1.1) in the half-band $(-\infty < x \leqslant 0, 0 \leqslant y \leqslant b)$, defining matrix B_3 as

$$B_3(\partial/\partial x,\ \partial/\partial y) \equiv \begin{Vmatrix} I & 0 \\ \mu\,\partial/\partial y & \mu\,\partial/\partial x \end{Vmatrix} \tag{1.11}$$

and stipulating boundedness of vector $\mathbf{U}(x, y)$ when $x \to -\infty$, the components $g_{ij}^n(x, \xi)$ of the kernel of integral (1.8) for $x \leqslant \xi$ are determined by formulas

$$g_{11}^n(x, \xi) = p(x - \xi) - q(x - \xi) - p(x - \xi) + q(x + \xi)$$

$$g_{12}^n(x, \xi) = m[p(x - \xi) + p(x + \xi)], \quad g_{21}^n(x - \xi) = p(x + \xi) - p(x - \xi)$$

$$g_{22}^n(x, \xi) = -m[p(x - \xi) + q(x - \xi) + p(x + \xi) + q(x + \xi)]$$

$$p(u) = {}^1/_4(\lambda + \mu)\,ue^{\nu u}, \quad q(u) = (4\nu)^{-1}(\lambda + 3\mu)\,e^{\nu u}$$

When $n = 0$ and $x \leqslant \varepsilon$ we have

$$g_{11}{}^{\circ} = m\mu\xi, \quad g_{12}{}^{\circ} = g_{21}{}^{\circ} = g_{22}{}^{\circ} = 0$$

Carrying out the summation in (1.10) and taking into account the known relationships

$$\sum_{n=1}^{\infty} t^n \cos n\gamma = (1 - t\cos\gamma)(1 - 2t\cos\gamma + t^2)^{-1} \tag{1.12}$$

$$\sum_{n=1}^{\infty} t^n n^{-1} \cos n\gamma = -\frac{1}{2}\ln(1 - 2t\cos\gamma + t^2)$$

$$t^2 < 1, 0 < \gamma < 2\pi$$

we obtain the expression for the element $G_{11}(x, y;\ \xi, \eta)$ of the sought Green's matrix in explicit form

$$G_{11}(x, y; \xi, \eta) = \frac{1}{2} m\mu \operatorname{Re}\zeta - \frac{1}{8\pi}(\lambda + 3\mu) b \ln \frac{E(z+\zeta)\ E(z+\bar{\zeta})}{E(z-\zeta)\ E(z-\bar{\zeta})} +$$

$$\frac{1}{8}(\lambda + \mu)\{\operatorname{Re}(z-\zeta)[Q(z-\zeta) + Q(z-\bar{\zeta})] -$$

$$\operatorname{Re}(z+\zeta)[Q(z+\zeta) + Q(z+\bar{\zeta})]\}, \quad E(u) = |1 - \omega(u)|,$$

$$Q(u) = P(u) E^{-2}(u)$$

$$\omega(u) = \exp[\pi b^{-1} u], \quad P(u) = \operatorname{Re}[1 - \omega(u)]$$

$$z = x + iy, \quad \zeta = \xi + i\eta$$

and for other elements of Green's matrix we have

$$G_{12}(x, y;\ \xi, \eta) = {}^1/_8\, \mu(\lambda + \mu)(\lambda + 2\mu)^{-1}\{\operatorname{Re}(z - \zeta)[T(z - \zeta) +$$

$$T(z - \bar{\zeta})] + \operatorname{Re}(z + \zeta)[T(z + \zeta) + T(z + \bar{\zeta})]\}$$

$$G_{21}(x, y;\ \xi, \eta) = {}^1/_8(\lambda + \mu)\{\operatorname{Re}(z - \zeta)[T(z - \zeta) - T(z - \bar{\zeta})] +$$

$$\operatorname{Re}(z + \zeta)[T(z + \zeta) - T(z + \bar{\zeta})]\}$$

$$G_{22}(x, y; \xi, \eta) = \frac{1}{8\pi}(\lambda + 3\mu) mb \ln \frac{E(z+\zeta)\, E(z-\bar{\zeta})}{E(z-\zeta)\, E(z+\bar{\zeta})} -$$

$$\frac{1}{8}(\lambda + \mu) m\{\operatorname{Re}(z-\zeta)[Q(z-\zeta) - Q(z-\bar{\zeta})] -$$

$$\operatorname{Re}(z+\zeta)[Q(z+\zeta) - Q(z+\bar{\zeta})]\}$$

$$T(u) = S(u)\ E^{-2}(u), \quad S(u) = \operatorname{Im}\omega(u)$$

The derived construction satisfies all properties that define Green's matrix.

2. We present below some of the results of application of the described method for deriving Green's functions of the Laplace operator for various boundary value and mixed problems. Thus, for instance, in the case of the Dirichlet problem for the half-band $(x \geqslant 0,\ 0 \leqslant y \leqslant b)$ the Green's function is expressed by the expansion

$$g(x, y; \xi, \eta) = 2b^{-1} \sum_{n=1}^{\infty} g_n(x, \xi) \sin \nu y \sin \nu\eta, \quad \nu = n\pi b^{-1}$$

$$g_n(x, \xi) = {}^1/_2 \nu^{-1} [\exp(-\nu(x + \xi)) - \exp(\nu(x - \xi))], \quad x \leqslant \xi$$

whose summation with the notation introduced in Sectn. 1 taken into account yields

$$g(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \frac{E(z - \bar{\zeta}) E(z + \bar{\zeta})}{E(z - \zeta) E(z + \zeta)} \tag{2.1}$$

Formula (2.1) coincides with that obtained in [2] using a different representation of Green's function for the half-band.

In the case of mixed problem

$$\partial v / \partial x \big|_{x=0} = 0, \quad v \big|_{y=0;\, b} = 0$$

by analogy to the above for the half-band we have

$$g(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \frac{E(z - \bar{\zeta}) E(z + \zeta)}{E(z - \zeta) E(z + \bar{\zeta})} \tag{2.2}$$

When the half-bands are considered with the condition

$$(\partial v / \partial x + \beta v)_{x=0} = 0, \quad v \big|_{y=0;\, b} = 0$$

we obtain

$$g(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \frac{E(z - \bar{\zeta}) E(z + \zeta)}{E(z - \zeta) E(z + \bar{\zeta})} - \tag{2.3}$$

$$2\beta b^{-1} \sum_{n=1}^{\infty} [\nu(\beta - \nu)]^{-1} \exp[-\nu(x + \xi)] \sin \nu y \sin \nu\eta, \quad \nu = n\pi b^{-1}$$

from which in the particular case of $\beta = 0$ we obtain formula (2.2).

Following the procedure described in Sect. 1 for the Dirichlet problem in the rectangle $(0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b)$ for the Green's function we obtain the following expression

$$g(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \frac{E(z - \bar{\zeta}) E(z + \bar{\zeta})}{E(z - \zeta) E(z + \zeta)} - \tag{2.4}$$

$$2b^{-1} \sum_{n=1}^{\infty} \left[\operatorname{sh} \nu x \operatorname{sh} \frac{\nu\xi}{\nu e^{\nu a}} \operatorname{sh} \nu a \right] \sin \nu y \sin \nu\eta$$

Mixed boundary value problems for a rectangle do not present any significant difficulties. Thus, when

$$v \big|_{x=0} = v \big|_{y=0;\, b} = (\partial v / \partial x + \beta v)_{x=a} = 0$$

we obtain

$$g(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \frac{E(z - \bar{\zeta}) E(z + \bar{\zeta})}{E(z - \zeta) E(z + \zeta)} - \tag{2.5}$$

$$2b^{-1}\sum_{n=1}^{\infty}\left[(\beta-\nu)\,\mathrm{sh}\,\nu x\,\mathrm{sh}\,\frac{\nu\xi}{\nu}\exp\nu a\,(\beta\,\mathrm{sh}\,a+\nu\,\mathrm{ch}\,\nu a)\right]\sin\nu y\sin\nu\eta$$

It will be readily seen that the functions specified by formulas (2.1)-(2.5) satisfy all of Green's functions properties.

3. We conclude by presenting some of the results of constructing Green's functions for mixed boundary value problems of the Laplace equation for regions whose boundaries are defined by coordinate lines in a system of polar coordinates. Thus for the Dirichlet's problem in the circular sector $(0\leqslant r\leqslant R,\ 0\leqslant\varphi\leqslant\alpha)$ Green's function is of the form

$$g(r,\varphi;\rho,\psi)=\frac{1}{2\pi}\ln\frac{|z^{\sigma}-\bar{\zeta}^{\sigma}|\,|(R^2z)^{\sigma}-(r^2\zeta)^{\sigma}|}{|z^{\sigma}-\zeta^{\sigma}|\,|(R^2z)^{\sigma}-(r^2\bar{\zeta})^{\sigma}|}\tag{3.1}$$

$$z=r(\cos\varphi+i\sin\varphi),\quad \zeta=\rho(\cos\psi+i\sin\psi),\ \sigma=\pi/\alpha$$

The problem

$$v|_{\varphi=0}=\partial v/\partial\varphi\,|_{\varphi=\alpha}=0$$

for sector $(0\leqslant\varphi\leqslant\alpha)$ yields

$$g(r,\varphi;\rho,\psi)=\frac{1}{2\pi}\ln\frac{|z^{\sigma/2}+\zeta^{\sigma/2}|\,|z^{\sigma/2}-\bar{\zeta}^{\sigma/2}|}{|z^{\sigma/2}-\zeta^{\sigma/2}|\,|z^{\sigma/2}+\bar{\zeta}^{\sigma/2}|}\tag{3.2}$$

which for $\alpha=\pi$ yields the expression for Green's function of the mixed boundary value problem for the half-plane.

In the case of an infinite ring sector $(r\geqslant R,\ 0\leqslant\varphi\leqslant\alpha)$ with boundary conditions

$$v|_{\varphi=0}=v|_{\varphi=\alpha}=\partial v/\partial r|_{r=R}=0$$

we obtain the following representation for Green's function

$$g(r,\varphi;\rho,\psi)=\frac{1}{2\pi}\ln\frac{|z^{\sigma}-\bar{\zeta}^{\sigma}|\,|(R^2z)^{\sigma}-(r^2\bar{\zeta})^{\sigma}|}{|z^{\sigma}-\zeta^{\sigma}|\,|(R^2z)^{\sigma}-(r^2\zeta)^{\sigma}|}\tag{3.3}$$

Green's function of the Laplace operator for the infinite ring sector with boundary conditions

$$v|_{\varphi=0}=v|_{r=R}=\partial v/\partial\varphi\,|_{\varphi=\alpha}=0$$

is of the form

$$g(r,\varphi;\rho,\psi)=\frac{1}{2\pi}\ln\left\{\frac{|z^{\sigma/2}+\zeta^{\sigma/2}|\,|z^{\sigma/2}-\bar{\zeta}^{\sigma/2}|}{|z^{\sigma/2}+\bar{\zeta}^{\sigma/2}|\,|z^{\sigma/2}-\zeta^{\sigma/2}|}\times\frac{|(\rho^2z)^{\sigma/2}+(R^2\bar{\zeta})^{\sigma/2}|\,|(\rho^2z)^{\sigma/2}-(R^2\zeta)^{\sigma/2}|}{|(\rho^2z)^{\sigma/2}+(R^2\zeta)^{\sigma/2}|\,|(\rho^2z)^{\sigma/2}-(R^2\bar{\zeta})^{\sigma/2}|}\right\}\tag{3.4}$$

Boundary conditions

$$v|_{\varphi=0;\,\alpha}=(\partial v/\partial r+\beta v)_{r=R}=0$$

for the infinite ring sector yield

$$g(r, \varphi; \rho, \psi) = \frac{1}{2\pi} \ln \frac{|z^\sigma - \bar{\zeta}^\sigma| \, |(R^2 z)^\sigma - (r^2 \bar{\zeta})^\sigma|}{|z^\sigma - \zeta^\sigma| \, |(R^2 z)^\sigma - (r^2 \zeta)^\sigma|} - \tag{3.5}$$

$$2\beta R \alpha^{-1} \sum_{n=1}^{\infty} [R^{2\nu} / \nu r^\nu \rho^\nu (\beta R - \nu)] \sin \nu\varphi \sin \nu\psi, \quad \nu = n\pi\alpha^{-1}$$

In the case of the boundary value problem

$$v|_{\varphi=0;\,\alpha} = v|_{r=R_1} = \partial v / \partial r|_{r=R_2} = 0$$

for the ring sector $(R_1 \leqslant r \leqslant R_2,\ 0 \leqslant \varphi \leqslant \alpha)$ we obtain Green's function of the form

$$g(r, \varphi; \rho, \psi) = \frac{1}{2\pi} \ln \frac{|z^\sigma - \bar{\zeta}\sigma| \, |(R_1{}^2 z)^\sigma - (r^2 \zeta)^\sigma|}{|z^\sigma - \zeta^\sigma| \, |(R_1{}^2 z)^\sigma - (r^2 \bar{\zeta})^\sigma|} + \tag{3.6}$$

$$\alpha^{-1} \sum_{n=1}^{\infty} \frac{(\rho^{2\nu} - R_1^{2\nu})(r^{2\nu} - R_1^{2\nu})}{\nu r \rho (R_1^{2\nu} + R_2^{2\nu})} \sin \nu\varphi \sin \nu\psi$$

Functions (3.1)-(3.6) satisfy all of conditions that determine Green's function.

The boundary value problems listed here for which Green's functions and matrices can be derived by the described method do not exhaust its application field. These problems should be viewed as an illustration of the effectiveness of the proposed method.

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Translated by J.J.D.